Theory of large rotations

Rotations

Rotations in three dimensions can be described in a number of different, but equivalent ways, for example the Euler angles, http://en.wikipedia.org/wiki/Euler_angles. Here we will use a vector representing the axis of rotation.

Consider a rotation is of magnitude α radians about the axis defined by the vector $\mathbf{a}$. A positive rotation is clockwise looking along $\mathbf{a}$. A negative rotation is anticlockwise, or clockwise looking in the opposite direction.

This definition means that the rotation causes the vector $\mathbf{u}$ to become the vector

$$\begin{aligned}\mathbf{u}' &= \frac{(\mathbf{u}\cdot\mathbf{a})\mathbf{a}}{a^2} + \left(\mathbf{u} - \frac{(\mathbf{u}\cdot\mathbf{a})\mathbf{a}}{a^2}\right)\cos\alpha + \frac{\mathbf{a}\times\mathbf{u}}{a}\sin\alpha \\ &= \mathbf{u} - \left(\mathbf{u} - \frac{(\mathbf{u}\cdot\mathbf{a})\mathbf{a}}{a^2}\right)(1-\cos\alpha) + \frac{\mathbf{a}\times\mathbf{u}}{a}\sin\alpha\end{aligned}$$

in which $a = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{\dfrac{\mathbf{A}:\mathbf{A}}{2}}$.

$\left(\mathbf{u} - \dfrac{(\mathbf{u}\cdot\mathbf{a})\mathbf{a}}{a^2}\right)$ is the component of $\mathbf{u}$ perpendicular to $\mathbf{a}$ in the plane of $\mathbf{u}$ and $\mathbf{a}$.

$\dfrac{(\mathbf{a}\times\mathbf{u})}{a}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{a}$.

Let us write $a = \tan\dfrac{\alpha}{2}$ (note returning to the original definition) since then

$$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = 2\tan\frac{\alpha}{2}\cos^2\frac{\alpha}{2} = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{2a}{1+a^2}$$

$$1-\cos\alpha = 1-\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2} = 2-2\cos^2\frac{\alpha}{2} = 2-\frac{2}{1+a^2} = \frac{2a^2}{1+a^2}$$

and

$$\mathbf{u}' = \mathbf{u} - \left(\mathbf{u} - \frac{(\mathbf{u}\cdot\mathbf{a})\mathbf{a}}{a^2}\right)\frac{2a^2}{1+a^2} + \frac{\mathbf{a}\times\mathbf{u}}{a}\frac{2a}{1+a^2} = \mathbf{u} + \frac{2}{1+a^2}\left[-\left(a^2\mathbf{u} - (\mathbf{u}\cdot\mathbf{a})\mathbf{a}\right) + \mathbf{a}\times\mathbf{u}\right].$$

Three rotation which have no net effect

Now let us consider a rotation $\mathbf{a}$ followed by a rotation $\mathbf{b}$ followed by a rotation $\mathbf{c}$ such that there is no net effect. Thus the rotation $-\mathbf{c}$ has the same effect as $\mathbf{a}$ followed $\mathbf{b}$.

We can permute so that $\mathbf{b}$ followed by $\mathbf{c}$ followed by $\mathbf{a}$ has no effect and the same applies to $\mathbf{c}$ followed by $\mathbf{a}$ followed by $\mathbf{b}$.

$\mathbf{c}$ has no effect upon itself and therefore $\mathbf{a}$ and $\mathbf{b}$ have opposite effects upon $\mathbf{c}$. Thus

$$\mathbf{c}+\frac{2}{1+a^2}\left[-\left(a^2\mathbf{c}-(\mathbf{c}\cdot\mathbf{a})\mathbf{a}\right)+\mathbf{a}\times\mathbf{c}\right]=\mathbf{c}+\frac{2}{1+b^2}\left[-\left(b^2\mathbf{c}-(\mathbf{c}\cdot\mathbf{b})\mathbf{b}\right)-\mathbf{b}\times\mathbf{c}\right]$$

$$\left(1+b^2\right)\left[a^2\mathbf{c}-(\mathbf{c}\cdot\mathbf{a})\mathbf{a}-\mathbf{a}\times\mathbf{c}\right]=\left(1+a^2\right)\left[b^2\mathbf{c}-(\mathbf{c}\cdot\mathbf{b})\mathbf{b}+\mathbf{b}\times\mathbf{c}\right]$$

$$b^2(\mathbf{c}\cdot\mathbf{a})-(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{b})+\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=0$$
$$a^2(\mathbf{b}\cdot\mathbf{c})-(\mathbf{c}\cdot\mathbf{a})(\mathbf{a}\cdot\mathbf{b})-\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})=0$$

Thus, using the scalar triple product,

$$b^2(\mathbf{c}\cdot\mathbf{a})-(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{b})+\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=0$$
$$a^2(\mathbf{b}\cdot\mathbf{c})-(\mathbf{c}\cdot\mathbf{a})(\mathbf{a}\cdot\mathbf{b})+\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=0$$

If $\mathbf{c}=\theta\mathbf{a}+\phi\mathbf{b}-\chi(\mathbf{a}\times\mathbf{b})$

$$b^2\left(\theta a^2+\phi(\mathbf{a}\cdot\mathbf{b})\right)-\left(\theta(\mathbf{a}\cdot\mathbf{b})+\phi b^2\right)(\mathbf{a}\cdot\mathbf{b})-\chi\left(a^2b^2-(\mathbf{a}\cdot\mathbf{b})^2\right)=0$$
$$a^2\left(\theta(\mathbf{a}\cdot\mathbf{b})+\phi b^2\right)-\left(\theta a^2+\phi(\mathbf{a}\cdot\mathbf{b})\right)(\mathbf{a}\cdot\mathbf{b})-\chi\left(a^2b^2-(\mathbf{a}\cdot\mathbf{b})^2\right)=0$$
$$\theta\left(a^2b^2-(\mathbf{a}\cdot\mathbf{b})^2\right)-\chi\left(a^2b^2-(\mathbf{a}\cdot\mathbf{b})^2\right)=0$$
$$\phi\left(a^2b^2-(\mathbf{a}\cdot\mathbf{b})^2\right)-\chi\left(a^2b^2-(\mathbf{a}\cdot\mathbf{b})^2\right)=0$$

so that $\theta=\phi=\chi$ and $\mathbf{c}=\chi\left[\mathbf{a}+\mathbf{b}-(\mathbf{a}\times\mathbf{b})\right]$, or, writing $\chi=-\frac{1}{1-C}$,

$$\mathbf{c}=-\frac{\mathbf{a}+\mathbf{b}-\mathbf{a}\times\mathbf{b}}{1-C}.$$

Therefore permuting,

$$
\begin{aligned}
\mathbf{a} &= -\frac{\mathbf{b}+\mathbf{c}-\mathbf{b}\times\mathbf{c}}{1-A} \\
&= -\frac{\mathbf{b}-\dfrac{\mathbf{a}+\mathbf{b}-\mathbf{a}\times\mathbf{b}}{1-C}-\mathbf{b}\times\left(\dfrac{\mathbf{a}+\mathbf{b}-\mathbf{a}\times\mathbf{b}}{1-C}\right)}{1-A} \\
&= -\frac{-C\mathbf{b}-\mathbf{a}+\mathbf{b}\times(\mathbf{a}\times\mathbf{b})}{(1-A)(1-C)}
\end{aligned}
$$

so that, scalar multiplying by $\mathbf{a}$,

$$
\begin{aligned}
a^2 &= -\frac{-C(\mathbf{a}\cdot\mathbf{b})-a^2+\mathbf{a}\cdot(\mathbf{b}\times(\mathbf{a}\times\mathbf{b}))}{(1-A)(1-C)} = -\frac{-C(\mathbf{a}\cdot\mathbf{b})-a^2-(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})}{(1-A)(1-C)} \\
&= -\frac{-C(\mathbf{a}\cdot\mathbf{b})-a^2-a^2b^2+(\mathbf{a}\cdot\mathbf{b})^2}{(1-A)(1-C)}
\end{aligned}
$$

or by $\mathbf{b}$,

$$
(\mathbf{a}\cdot\mathbf{b}) = -\frac{-Cb^2-(\mathbf{a}\cdot\mathbf{b})}{(1-A)(1-C)}.
$$

Thus

$$
(\mathbf{a}\cdot\mathbf{b})\left(-C(\mathbf{a}\cdot\mathbf{b})-a^2-a^2b^2+(\mathbf{a}\cdot\mathbf{b})^2\right) = a^2\left(-Cb^2-(\mathbf{a}\cdot\mathbf{b})\right)
$$

and therefore

$$
C = \frac{(\mathbf{a}\cdot\mathbf{b})\left(-a^2-a^2b^2+(\mathbf{a}\cdot\mathbf{b})^2\right)+a^2(\mathbf{a}\cdot\mathbf{b})}{(\mathbf{a}\cdot\mathbf{b})^2-a^2b^2} = \mathbf{a}\cdot\mathbf{b}\ .
$$

Thus $\mathbf{c} = -\dfrac{\left[\mathbf{a}+\mathbf{b}-(\mathbf{a}\times\mathbf{b})\right]}{1-\mathbf{a}\cdot\mathbf{b}}$.

Note the minus sign because we are assuming that the rotation $-\mathbf{c}$ has the same effect as $\mathbf{a}$ followed $\mathbf{b}$.

This is the same result that can be obtained using quaternions.